(Tutorial 3) :

Ex1. Let $A \in \mathbb{R}$. define $f_1 : [0;\infty) \longrightarrow \mathbb{R}$ by

$$f_1(x) := \begin{cases} \sin\frac{1}{x} & \text{for } x \in (0;\infty) \\ A & \text{for } x = 0 \end{cases}$$

Then, f_1 is not continuous at 0.

Ex2: define $f_2 : [0,\infty) \longrightarrow \mathbb{R}$ by $f_2(x) := \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Then, f_2 is cont at 0

Ex3: Show that the length of curve f_2 on $[0,1]$ is ∞.

(Ans)

Ex I: $\forall \ \delta > 0$, $\exists \ x_1, x_2 \in (0,\delta)$ s.t. $f_1(x_1) = 0$, $f_1(x_2) = 1$

So, $\lim_{x\to 0} f_1(x)$ does not exist.

In other words, Suppose f_1 is cont at 0; i.e. $\lim_{x\to 0} f_1(x) = \lim_{x\to 0} \sin\frac{1}{x} = A$.

Consider a sequence $y_n = \frac{1}{2\pi n}$, then $y_n \to 0$ and $A = \lim_{x\to 0} f_1(x) = \lim_{n\to\infty} f_1(y_n) = 0$

Consider another sequence $z_n = \frac{1}{\frac{\pi}{2}(2n+1)}$, then $z_n \to 0$

and $0 = A = \lim_{n\to\infty} f_1(z_n) = \lim_{n\to\infty} (-1)^n$. But $\lim_{n\to\infty} (-1)^n$ does not exist.

Ex2: For $x \neq 0$,

$$-x \leq x\sin\frac{1}{x} = f_2(x) \leq x$$

Both L.H.S and R.H.S tend to 0 as x tends to 0.

$\therefore \lim_{x\to 0} f_2(x) = 0 = f(0)$ i.e. f is cont at 0.

Ex3: Procedures:

① We divide $[0,1]$ into subintervals I_n s.t. they are disjoint from each other (e.g. $I_n := [\frac{1}{\frac{\pi}{2}(2n+1)}, \frac{1}{\frac{\pi}{2}(2n)}]$) and find a lower bound c_n for the length of curve f_2 on I_n.

② For any $N \in \mathbb{N}$,
Length of curve f_2 on $[0,1] \geq$ length of curve f_2 on $\bigcup_{n=1}^{N} I_n$

$$\geq \sum_{n=1}^{N} c_n$$

and $\sum_{n=1}^{N} c_n \to \infty$ as $N \to \infty$

Ex 3: (Cont):

For Procedure ①: Let: $I_n := [\frac{1}{\frac{\pi}{2}(2n+1)}, \frac{1}{\frac{\pi}{2}(2n)}]$

$$f_2\left(\frac{1}{\frac{\pi}{2}(2n+1)}\right) = \frac{1}{\frac{\pi}{2}(2n+1)}(-1)^n \quad , \quad f_2\left(\frac{1}{\frac{\pi}{2}(2n)}\right) = 0$$

The curve f_2 on I_n is connecting two points $(x,y) = \left(\frac{1}{\frac{\pi}{2}(2n+1)}, \frac{1}{\frac{\pi}{2}(2n+1)}(-1)^n\right)$ and $(x,y) = \left(\frac{1}{\frac{\pi}{2}(2n)}, 0\right)$.

Since the line segment connecting these two points is the shortest curve connecting these two points,

length of f_2 on I_n $\geq$ length of the line segment $\underset{\uparrow}{\geq}$ $\frac{1}{\frac{\pi}{2}(2n+1)} = \frac{2}{\pi}\frac{1}{2n+1} \geq \frac{1}{\pi}\cdot\frac{1}{n+1}$

Pythagorean thm

Take the lower bound $C_n = \frac{1}{\pi}\frac{1}{n+1}$

Procedure ②: $\sum_{n=1}^{N} C_n = \frac{1}{\pi}\sum_{n=1}^{N}\frac{1}{n+1} \to \infty$ as $N \to \infty$.